Fwtree
A GAP4 Package

Version 1.3

by

Bettina Eick
email: beick@tu-bs.de

and

Tobias Rossmann
email: tobias.rossmann@googlemail.com

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1 Introduction

This package provides GAP-functions to reproduce the experimental results described in our paper [ER09].

More precisely, it provides

- functions to determine the rank, width and obliquity of a finite p-group,
- functions to investigate the graph of all finite p-groups of a given rank, width and obliquity using the ANUPQ-package [ONG06], and
- a library of finite quotients of certain infinite pro-p-groups of finite rank, width and obliquity.

2 Methods and functions

This chapter describes all the main methods and functions of this package.

2.1 Functions for finite p-groups

Let G be a finite p-group given by a consistent polycyclic presentation as Pc group.

1 ► `LCSFactorTypes( G )`

returns the abelian invariants of the lower central series factors of G.

2 ► `LCSFactorSizes( G )`

returns the orders of the lower central series factors of G.

3 ► `WidthPGroup( G )`

returns the width of G.

4 ► `SubgroupRank( G )`

returns the (subgroup-)rank of G.

5 ► `Obliquity( G )`

returns the obliquity of G.

6 ► `HasObliquityZero( G )`

checks whether G has obliquity 0 and returns true or false.

2.2 Functions to generate groups and trees

Let $G(p, rwo)$ denote the full tree of all finite p-groups with rank $rwo[1]$, width $rwo[2]$ and obliquity $rwo[3]$. This tree can be finite or infinite; if it is infinite, then the infinite pro-p-groups of the considered rank, width and obliquity specify infinite subtrees of the full tree. The groups not contained in such an infinite subtree are called sporadic.

1 ► `GroupsByRankWidthObliquity( p, d, rwo, roots, limit )`

determines all p-groups G with $G/\Phi(G)$ of order p^d and rank, width and obliquity as prescribed in *rwo* up to order *limit*. Here p and d are integers, *rwo* is a list of three integers and *limit* is an integer.

The parameter *roots* is a list of groups described by their id's with respect to the small groups library. The descendants of the groups described in *roots* are excluded from the output of this function. This option can be used to prune the tree of groups determined by this function.

If there are only finitely many sporadic p-groups with given rank, width and obliquity, then this function can be used to generate them; in this case *roots* must contain a complete list of all id's of roots of infinite subtrees and *limit* can be set to infinity.

2 ▶ `BranchRWO( G, i, rwo )`

for a stable quotient (see [ER09]) G of a pro-p-group of rank $rwo[1]$, $rwo[2]$ and obliquity $rwo[3]$, this function returns the i-th branch of its corresponding tree. The structure of the tree is encoded in a list. If one of the global parameters `CHECK_RANK` or `CHECK_OBLIQUITY` is set to false, then checking the corresponding invariant is omitted and hence a potentially larger tree is returned.

The user is advised not to perform any other computations using ANUPQ or the pq-program while using this or the following function, because such computations will be terminated.

3 ▶ `BoundedDescendantsRWO( G, i, c, rwo)`

returns the tree of all descendants of $G/\gamma_i(G)$ of rank $rwo[1]$, width $rwo[2]$, obliquity $rwo[3]$ and class at most c.

4 ▶ `DrawBranch( branch )`

if the package is run under XGap, then this function can be used to draw a branch as output by the above two functions in the case of width 2. The user may wish to improve the quality of the output by modifying the file `gap/xbranch.gi`.

Vertices drawn on the same level correspond to groups of the same class. If G is a descendant of H in the branch, then G is drawn as a filled circle if $|G| = |H|p$ and as a solid box if $|G| = |H|p^2$.

The package also provides finite quotients of a number of infinite pro-p-groups with finite rank, width and obliquity. Throughout the section, p is an odd prime.

5 ▶ `ProPSylowGroupOfPSL(d, p, n)`

returns the quotient of the Sylow pro-p-subgroup of $\mathrm{PSL}_d(\mathbb{Q}_p)$ modulo the matrices which are congruent to the identity modulo p^n.

6 ▶ `ProPSylowGroupOfPSF(p, n)`

Let L be the simple Lie algebra of dimension 3 over $\mathbb{Q}_p$ which is not isomorphic to $sl_2(\mathbb{Q}_p)$. This function returns a finite quotient of the Sylow pro-p-subgroup of its automorphism group. The parameter n specifies how large this quotient is.

In [KLGP97], a library of maximal pro-p-groups with finite rank, width and obliquity corresponding to the Lie algebras of small dimension is provided. Here, we provide a library of large quotients of these groups for some of the Lie algebras of type $sl_d(K)$, where K is a finite extension of $\mathbb{Q}_p$. These groups have been determined using the programs described in [KLGP97]. To be precise, depending on the group, it may be necessary to pass to the quotient by one of the last non-trivial terms of the lower central series in order to obtain a quotient of the respective pro-p-group.

7 ▶ `ProPQuotient(p, dim, deg, no)`

returns a finite group corresponding to the maximal pro-p-group G with Lie algebra $sl_{dim}(K)$, where K is a field of degree deg over $\mathbb{Q}_p$. The parameter no specifies the number of the group in our database.

2.3 Example

When run under XGap, the following code constructs and draws the branch with root $G/\gamma_5(G)$ in the graph of finite 5-groups of rank 3, width 2 and obliquity 0, where G is the Sylow pro-p-subgroup of $\mathrm{Aut}(sl_2(\mathbb{Q}_5))$.

```
gap> g := ProPSylowGroupOfPSL(2,5,6);
Pcp-group with orders [ 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5 ]
gap> branch := BranchRWO(g,5,[3,2,0]);;
ConstructBranch: root-p-class: 4
Constructed 3 1-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
Constructed 3 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
gap> DrawBranch(branch);
```

A window with the following graph should appear.

Bibliography

[ER09] Bettina Eick and Tobias Rossmann. Periodicities for graphs of p-groups beyond coclass. 2009. Preprint.

[KLGP97] G. Klaas, C. R. Leedham-Green, and W. Plesken. *Linear pro-p-groups of finite width*, volume 1674 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[ONG06] Eamonn O'Brien, Werner Nickel, and Greg Gamble. *ANUPQ — A GAP4 Package, Version 3.0*, 2006.

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This index covers only this manual. A page number in *italics* refers to a whole section which is devoted to the indexed subject. Keywords are sorted with case and spaces ignored, e.g., "PermutationCharacter" comes before "permutation group".